R-matrix and covariant q-superoscillators for $U_q(gl(1 \bmod 1))$

1992 J. Phys. A: Math. Gen. 25 L1159

(http://iopscience.iop.org/0305-4470/25/19/005)

J. Phys. A: Math. Gen. 25 (1992) L1159–L1165. Printed in the UK

LETTER TO THE EDITOR

$\boldsymbol{R}$-matrix and covariant q-superoscillators for $\mathcal{U}_q(\mathrm{gl}(1|1))$

Jean-Christophe Wallet

Division de Physique Théorique†, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France

Received 21 June 1992

Abstract. We present a construction, based on an R-matrix formalism, of covariant quantum superoscillator algebras for $\mathcal{U}_q(\mathrm{gl}(1|1))$. We show that the complete structure of the oscillator algebras stems from a suitable combination of properties of the $\hat{R}$-matrix, covariance of the oscillators at the deformed level and associativity.

Numerous systems of (bosonic and/or fermionic) q-oscillators have been proposed and used to construct realizations of quantum (super)algebras (see e.g. [1]). Most of these studies proceed in the spirit of the initial analysis of [2, 3].

However, a natural scheme which preserves the covariance of the oscillators at the deformed level has been presented in [4] for $\mathrm{SU}_q(n)$. This second approach can also be translated into the R-matrix formalism, a fact that has been exploited intensively for the quantum universal enveloping algebra $\mathcal{U}_q(\mathrm{sl}(2))$ in the analysis of [5], more directed towards applications to rational conformal field theory.

The later R-matrix approach can also be applied to quantum universal enveloping super-algebras in view of the obtention of covariant q-superoscillator algebras.

In the main part of this letter, we perform such a construction in the simple case of $\mathcal{U}_q(\mathrm{gl}(1|1))$ for which the main steps of the algorithm can be easily followed. These steps are summarized beforehand together with some properties of the $\mathrm{gl}(1|1)$ super-algebra. The q-superoscillator algebras we find coincide with those derived in [6] using a somewhat different approach. Their origin is more transparent within the present construction.

The algorithm we use in this work can be applied to more complicated situations, in particular to $\mathcal{U}_q(\mathrm{osp}(1,2))$ and $\mathcal{U}_q(\mathrm{osp}(2,2))$‡. The corresponding analysis will appear elsewhere [7].

The $\mathrm{gl}(1|1)$ superalgebra (see [8] for properties of (classical) superalgebras) involves two even and two odd generators, denoted respectively by h, z and e, f. The following commutation relations hold

$$[z,e]=[z,f]=[z,h]=0 \qquad [h,e]=e \qquad [h,f]=-f \qquad \{e,f\}=z \qquad e^2=f^2=0. \tag{1}$$

† Unité de Recherche des Universités Paris 11 et Paris 6 associée au CNRS.

‡ These algebras are expected to be related to $N=1$ and $N=2$ superconformal theories.

In what follows, $d(\cdot)$ denotes a Z_2-grading. An object x is said to be even (odd) if one has $d(x)=0(1)$. One has the following useful formula: $(a\otimes b)(c\otimes d)=(-1)^{d(b)d(c)}(ac)\otimes(bd)$. Recall that gl$(1|1)$ is completely solvable; all its finite-dimensional irreducible representations are two dimensional† [8]. They are labelled by a pair of parameters (α,β). The generators can then be written as

$$e=\begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}\qquad f=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\qquad z=\begin{pmatrix}\beta & 0\\ 0 & \beta\end{pmatrix}\qquad h=\begin{pmatrix}\alpha & 0\\ 0 & \alpha-1\end{pmatrix}. \tag{2}$$

By convention, the corresponding basis vectors $(|e_1\rangle,|e_2\rangle)$ satisfy $d(|e_1\rangle)=0$, $d(|e_2\rangle)=1$. In the following, we choose for definiteness $(\alpha,\beta)=(\frac{1}{2},1)$. The extension of the subsequent analysis to arbitrary values is obvious and does not alter the conclusions as it will be shown.

The Jordan–Bargmann–Schwinger type construction of an oscillator model for gl$(1|1)$ is straightforward. One introduces a pair of creation (resp. annihilation) operators $(\bar{a},\bar{b})$ (resp. (a,b)), transforming under the $(\alpha,\beta)=(\frac{1}{2},1)$ representation of gl$(1|1)$ (resp. transposed $(-\frac{1}{2},1)$). One has $d(\bar{a})=d(a)=0$, $d(\bar{b})=d(b)=1$. Then, one defines four bilinear $T^{(M)}$ in the creation and annihilation operators, generically written as

$$T^{(M)}=(\bar{a},\bar{b})M\begin{pmatrix}a\\ b\end{pmatrix}$$

($M=e,f,h,z$ given in (2) with $\alpha=\frac{1}{2}$, $\beta=1$). The four $T^{(M)}$ are easily found to satisfy the commutation relations for gl$(1|1)$ given in (1), provided $[a,\bar{a}]=\{b,\bar{b}\}=1$, $[a,\bar{b}]=[\bar{a},b]=[\bar{a},\bar{b}]=[a,b]=b^2=\bar{b}^2=0$.

We decided to extend the above construction to $\mathcal{U}_q(\mathrm{gl}(1|1))\equiv\mathcal{U}_q$ in such a way that covariance of the oscillators is preserved at the deformed level. This involves three basic ingredients:

(i) The knowledge of the (braid) $\hat{R}$-matrix, built from the R-matrix.

(ii) The requirement that the quadratic products of the creation operators transform under a given irreducible representation of $\mathcal{U}_q$ appearing in the tensor product of two $(\frac{1}{2},1)$ representations or, equivalently, that they are eigenvectors spanning a given eigensubspace of $\hat{R}$ acting in $\rho\otimes\rho$. In what follows, $\rho\equiv\rho_{(1/2,1)}$ is the two-dimensional $\mathcal{U}_q$-module corresponding to $(\alpha,\beta)=(\frac{1}{2},1)$.

The condition (ii) fixes the q-deformed commutation relations among $\bar{a}$ and $\bar{b}$, whereas those for the annihilation operators a and b are obtained by transposition.

(iii) The requirement of associativity among the mixed products of q-oscillators finally leads to two distinct q-superoscillator algebras.

Within the present construction, the $\mathcal{U}_q$-covariance properties of the q-oscillators are induced by the coproduct with which $\mathcal{U}_q$ is equipped. Notice that in the approach of [2, 3], the $\mathcal{U}_q$-transformation properties of the q-oscillators are lost.

The relevant commutation relations for $\mathcal{U}_q$ are

$$\begin{aligned}&\{e,f\}=\frac{q^z-q^{-z}}{q-q^{-1}}\equiv[z]_q\qquad q^ze=eq^z\qquad q^zf=fq^z\\ &q^he=eq^{h+1}\qquad q^hf=fq^{h-1}.\end{aligned}\tag{3}$$

In the following, q is *generic* (i.e. not a root of unity). Then, the irreducible representations of $\mathcal{U}_q$ are in one-to-one correspondence with those of gl$(1|1)$.

† Apart from the trivial one-dimensional representation.

Let $|e_1\rangle$, $|e_2\rangle$ be the basis vectors spanning the two-dimensional $\mathcal{U}_q$-module ρ, with $d(|e_1\rangle)=0$, $d(|e_2\rangle)=1$. One has

$$\begin{array}{llll} q^h|e_1\rangle = q^{1/2}|e_1\rangle & q^z|e_1\rangle = q|e_1\rangle & e|e_1\rangle = 0 & f|e_1\rangle = |e_2\rangle \\ q^h|e_2\rangle = q^{-1/2}|e_2\rangle & q^z|e_2\rangle = q|e_2\rangle & e|e_2\rangle = |e_1\rangle & f|e_2\rangle = 0. \end{array} \tag{4}$$

The associative algebra $\mathcal{U}_q$ is equipped with a coproduct Δ, an antipode γ and a co-unit ε, respectively defined by

$$\begin{array}{ll} \Delta: & \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q \qquad \Delta(h) = h\otimes 1 + 1\otimes h \\ & \Delta(z) = z\otimes 1 + 1\otimes z \qquad \Delta(e) = q^{(z-h)/2}\otimes e + e\otimes q^{(-h-z)/2} \\ & \Delta(f) = q^{(h+z)/2}\otimes f + f\otimes q^{(h-z)/2} \end{array} \tag{5}$$

$$\begin{array}{ll} \gamma: & \mathcal{U}_q \to \mathcal{U}_q \qquad \gamma(q^h) = q^{-h} \qquad \gamma(q^z) = q^{-z} \\ & \gamma(e) = -q^{(h+z)/2} e q^{(h-z)/2} \qquad \gamma(f) = -q^{(z-h)/2} f q^{(-h-z)/2} \end{array} \tag{6}$$

$$\varepsilon: \quad \mathcal{U}_q \to C \qquad \varepsilon(h) = \varepsilon(z) = \varepsilon(e) = \varepsilon(f) = 0. \tag{7}$$

These three algebra homomorphisms satisfy, for all $x, y \in \mathcal{U}_q$:

(a1) $(id\otimes\Delta)\Delta(x) = (\Delta\otimes id)\Delta(x)$

(a2) $(\varepsilon\otimes id)\Delta(x) = (id\otimes\varepsilon)\Delta(x) = x$

(a3) $m(id\otimes\gamma)\Delta(x) = m(\gamma\otimes id)\Delta(x) = \varepsilon(x)1$, where m is the associative multiplication m: $\mathcal{U}_q\otimes\mathcal{U}_q \to \mathcal{U}_q$, $m(x\otimes y) = xy$.

There exists furthermore an invertible R-matrix, $R\in\mathcal{U}_q\otimes\mathcal{U}_q$, verifying

(b1) $(\sigma\circ\Delta)R = R\Delta$, with $\sigma(x\otimes y) = (-1)^{d(x)\,d(y)} y\otimes x$,

(b2) $(\Delta\otimes id)R = R_{12}R_{23}$,

(b3) $(id\otimes\Delta)R = R_{13}R_{12}$,

(b4) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (graded Yang–Baxter equation),

in usual notations. This ensures that $\mathcal{U}_q$ is a quasi-triangular Hopf algebra.

Using the coproduct (5), together with (4) and property (b1), the expression for the R-matrix acting in $\rho\otimes\rho$, is easily found to be

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \tag{8}$$

in the basis $|e_1e_1\rangle$, $|e_1e_2\rangle$, $|e_2e_2\rangle$ ($|e_ie_j\rangle \equiv |e_i\rangle\otimes e_j\rangle$).

Let $\mathcal{P}$ be the (graded) permutation matrix, generically defined by $\mathcal{P}(|e_i\rangle\otimes|e_j\rangle) = (-1)^{d(e_i)\,d(e_j)}|e_j\rangle\otimes|e_i\rangle$. From (8), the (braid) matrix $\hat{R} = \mathcal{P}R$ acting in $\rho\otimes\rho$ is given by

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & & 0 & -q^{-1} \end{pmatrix} \tag{9}$$

and satisfies $\Delta\hat{R} = \hat{R}\Delta$.

$\hat{R}$ admits two doubly degenerate eigenvalues, $+q$ and $-q^{-1}$. The corresponding eigenvectors are given by

$$(+q) \qquad \left[|e_1e_1\rangle; \frac{q^{1/2}|e_1e_2\rangle + q^{-1/2}|e_2e_1\rangle}{\sqrt{[2]_q}}\right] \tag{10a}$$

$$(-q^{-1}) \qquad \left[\frac{q^{-1/2}|e_1e_2\rangle - q^{1/2}|e_2e_1\rangle}{\sqrt{[2]_q}}; |e_2e_2\rangle\right]. \tag{10b}$$

They are respectively associated with the '$\mathcal{U}_q$-symmetric' and '$\mathcal{U}_q$-antisymmetric' (two-dimensional) irreducible parts of the orthogonal decomposition

$$\rho \otimes \rho = \rho_{(1,2)}^{(\mathrm{sym},+q)} \oplus \rho_{(0,2)}^{(\mathrm{asym},-q^{-1})}. \tag{11}$$

Let $|0\rangle$ and $\langle 0|$ be the ket and bra Fock vacua, assumed to be $\mathcal{U}_q$-invariant, that is, for all $x \in \mathcal{U}_q$, $x|0\rangle = \varepsilon(x)|0\rangle$, $\langle 0|x = \langle 0|\varepsilon(x)$, with ε defined in (7).

One introduces a pair of creation operators, $(\bar{a}, \bar{b})$, with $d(\bar{a}) = 0$, $d(\bar{b}) = 1$; $\bar{a}$ (resp. $\bar{b}$) creates a state $|e_1\rangle$ (resp. $(|e_2\rangle)$. $(\bar{a}, \bar{b})$ transforms under the irreducible two-dimensional $(\alpha, \beta) = (\frac{1}{2}, 1)$ representation of $\mathcal{U}_q$. The corresponding $\mathcal{U}_q$-covariance properties are induced by the coproduct (5); they will be listed below (see (16), (17)).

Now the commutation relations among $\bar{a}$ and $\bar{b}$ are fixed by the irreducibility requirement (ii): the quadratic products in $\bar{a}$ and $\bar{b}$ must transform under the '$\mathcal{U}_q$-symmetric' irreducible part $\rho_{(1,2)}^{(\mathrm{sym},+q)}$ in (11). This last choice is dictated by the fact that the q-oscillator algebras to be constructed involve simultaneously even and odd oscillators and that they must reduce to the undeformed case when $q = 1$. The condition (ii) requires that $(\bar{a}\bar{a}, \bar{a}\bar{b}, \bar{b}\bar{a}, \bar{b}\bar{b})$ is an eigenvector of $\hat{R}$ with eigenvalue $+q$. Then, using (9), we obtain

$$\bar{a}\bar{b} - q\bar{b}\bar{a} = 0 \qquad \bar{b}^2 = 0. \tag{12}$$

A similar analysis gives, after transposition, the commutation relations for the annihilation operators (a, b). We get

$$ab - q^{-1}ba = 0 \qquad b^2 = 0. \tag{13}$$

Equations (12) (resp. (13)), acting on $|0\rangle$ (resp. $\langle 0|$) imply the vanishing of the Fock space realization of the (resp. transposed) $\mathcal{U}_q$-antisymmetric vectors (10*b*).

Finally, the requirement of associativity among the mixed products of oscillators fixes the remaining commutation relations. After some calculations, we obtain two distinct q-superoscillator algebras given by

Solution (I): equations (12), (13)

$$\begin{aligned} &a\bar{b} - q^{-1}\bar{b}a = 0 \qquad \bar{a}b - qb\bar{a} = 0 \\ &a\bar{a} - q^{-2}\bar{a}a = 1 \\ &b\bar{b} + \bar{b}b = 1 - (1 - q^{-2})\bar{a}a \end{aligned} \tag{14}$$

Solution (II): equations (12), (13)

$$\begin{aligned} &a\bar{b} - q\bar{b}a = 0 \qquad \bar{a}b - q^{-1}b\bar{a} = 0 \\ &a\bar{a} - q^2\bar{a}a = 1 - (1 - q^2)\bar{b}b \\ &b\bar{b} + \bar{b}b = 1. \end{aligned} \tag{15}$$

Some comments are in order.

It can be pointed out that the key of the present construction is the matrix $\hat{R} = \mathcal{P}R$ acting in $\rho \otimes \rho$. It can be practically determined from the coproduct (5). Then, the irreducibility requirement (ii) fixes the commutation relations among creation or annihilation operators which permit one to obtain the complete q-superoscillator algebras by further demanding associativity.

The relations (14), (15) coincide with those proposed in [6]. Their origin is made more transparent within the present algorithm. Notice also that the algorithm can be applied to more complicated situations [7], once the corresponding $\hat{R}$-matrix acting in the tensor product of two (actually fundamental) irreducible representations is determined.

The structure of the coproduct Δ in (5) determines the $\mathcal{U}_q$-transformations of the creation operators $\bar{a}$ and $\bar{b}$. These latter act as linear mappings from $\mathcal{V}$, the set of tensor products of irreducible $\mathcal{U}_q$-modules, into itself: for any $|v\rangle \in \mathcal{V}$, one has $\bar{a}|v\rangle = |e_1\rangle \otimes |v\rangle$, $\bar{b}|v\rangle = |e_2\rangle \otimes |v\rangle$. Their natural transformations induced by the coproduct (5) are given by

$$\begin{gathered} q^h\bar{a} = \bar{a}q^{h+1/2} \qquad q^z\bar{a} = \bar{a}q^{z+1} \qquad q^h\bar{b} = \bar{b}q^{h-1/2} \qquad q^z\bar{b} = \bar{b}q^{z+1} \\ e\bar{a} = q^{1/4}\bar{a}e \qquad f\bar{a} = q^{3/4}\bar{a}f + \bar{b}q^{(h-z)/2} \\ e\bar{b} = -q^{3/4}\bar{b}e + \bar{a}q^{(-h-z)/2} \qquad f\bar{b} = -q^{1/4}\bar{b}f \end{gathered} \tag{16}$$

and the $\mathcal{U}_q$-transformations for the annihilation operators are

$$\begin{gathered} q^h a = aq^{h-1/2} \qquad q^z a = aq^{z-1} \qquad q^h b = bq^{h+1/2} \qquad q^z b = bq^{z-1} \\ ea = q^{-3/4}ae - bq^{(h-z)/2} \qquad fa = q^{-1/4}af \\ eb = -q^{-1/4}be \qquad fb = -q^{-3/4}bf + aq^{(-h-z)/2}. \end{gathered} \tag{17}$$

The transformations (17) are related to (16) by the (anti-involution) transposition defined by

$$\tilde{e} = f \qquad \tilde{f} = -e \qquad \tilde{\bar{a}} = a \qquad \tilde{\bar{b}} = -b \qquad \tilde{a} = \bar{a} \qquad \tilde{b} = \bar{b}. \tag{18}$$

Notice that the algebras (14), (15) are both invariant under $\tilde{\bar{a}} = a$, $\tilde{a} = \bar{a}$, $\tilde{\bar{b}} = -b$, $\tilde{b} = \bar{b}$. It can be verified that (12), (13) are invariant under (16), (17) whereas the mixed commutation relations in (14), (15) are not. These latter, however, are invariant under transformations induced by a coproduct Δ' related to Δ (5) by $\Delta' = \Delta_{|h=0}$ which can equip an $\mathcal{U}_q(\mathrm{sl}(1|1))$ subalgebra. The transformations are given by

$$\begin{gathered} q^z\bar{a} = \bar{a}q^{z+1} \qquad q^z\bar{b} = \bar{b}q^{z+1} \qquad q^z a = aq^{z-1} \qquad q^z b = bq^{z-1} \\ e\bar{a} = q^{1/2}\bar{a}e \qquad ea = q^{-1/2}ae - bq^{-z/2} \\ f\bar{a} = q^{1/2}\bar{a}f + \bar{b}q^{-z/2} \qquad fa = q^{-1/2}af \\ e\bar{b} = -q^{1/2}\bar{b}e + \bar{a}q^{-z/2} \qquad eb = -q^{-1/2}be \\ f\bar{b} = -q^{1/2}\bar{b}f \qquad fb = -q^{-1/2}bf + aq^{-z/2} \end{gathered} \tag{19}$$

and leave invariant the quadratic expression

$$\mathcal{X} = \bar{a}a + \bar{b}b. \tag{20}$$

Using an R-matrix formalism, we have presented the construction of covariant q-superoscillators algebras for $\mathcal{U}_q(\mathrm{gl}(1|1))$. Once the $\hat{R}$-matrix acting in $\rho \otimes \rho$ is known, the irreducibility requirement of the quadratic products of creation or annihilation operators, combined with the requirement of associativity of the mixed products single out in each case two distinct q-oscillator algebras. The covariance properties of the oscillators are determined by the coproduct of the considered quantum universal enveloping algebra.

In the one-parameter case, the two oscillator algebras we found coincide with those obtained in [6]. Our present approach makes their origin more transparent since it exhibits clearly the properties which lead to the various commutation relations. The present algorithm extends, rather easily, to more complicated superalgebras. In any case, the $\mathcal{U}_q$-covariance properties of the resulting systems of q-oscillators plays a key role in applications to statistical models or to superconformal theories [7].

According to the properties of gl(1 | 1), the above analysis could have been performed with creation (resp. annihilation) operators transforming under any irreducible ($\alpha, \beta \neq 0$) representation (resp. transposed). Then, using

$$
\begin{aligned}
&q^h|e_1\rangle = q^\alpha|e_1\rangle \qquad q^z|e_i\rangle = q^\beta|e_i\rangle, \mathrm{i}=1,2 \qquad e|e_1\rangle = 0 \qquad f|e_1\rangle = |e_2\rangle \\
&q^h|e_2\rangle = q^{\alpha-1}|e_2\rangle \qquad e|e_2\rangle = [\beta]_q|e_1\rangle \qquad f|e_2\rangle = 0
\end{aligned}
\tag{21}
$$

where $|e_i\rangle$, $i=1,2$ are now basis vectors of the irreducible 2D $\mathcal{U}_q$-module $\rho_{(\alpha,\beta)}$, together with (5) and (b1), the expressions for R and $\hat{R}$ both acting in $\rho_{(\alpha,\beta)} \otimes \rho_{(\alpha,\beta)}$ are found to be

$$
R = \begin{pmatrix} q^\beta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (q^\beta - q^{-\beta}) & 1 & 0 \\ 0 & 0 & 0 & q^{-\beta} \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} q^\beta & 0 & 0 & 0 \\ 0 & (q^\beta - q^{-\beta}) & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-\beta} \end{pmatrix}. \tag{22}
$$

R and $\hat{R}$ do not depend on α. It can be easily seen that $\hat{R}$ admits two doubly degenerate eigenvalues q^β and $-q^{-\beta}$. Then, selecting again the eigensubspace with eigenvalue q^β, the condition (ii) (together with transposition) yields

$$
\bar{a}\bar{b} - q^\beta \bar{b}\bar{a} = 0 \qquad ab - q^{-\beta} ba = 0 \qquad \bar{b}^2 = b^2 = 0 \tag{23}
$$

from which the final requirement of associativity leads to two q-superoscillator algebras which can be obtained from (14), (15) just by changing q into q^β. Again, the $\mathcal{U}_q$-covariance properties of the oscillators can be deduced from the action of (5).

Finally, we note that one can obtain a possible two-parameter extension of the results derived in this letter from a recently proposed two-parameter deformation of the universal enveloping algebra of gl(1 | 1) [9]. The relevant R-matrix together with the $\hat{R}$-matrix are [9]

$$
R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - p^{-1} & qp^{-1} & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix} \tag{24a}
$$

$$
\hat{R} \equiv \mathcal{P} R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - p^{-1} & qp^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -p^{-1} \end{pmatrix}. \tag{24b}
$$

$\hat{R}$ admits two doubly degenerate eigenvalues, $-p^{-1}$ and $+q$.

Requiring again that $(\bar{a}\bar{a}, \bar{a}\bar{b}, \bar{b}\bar{a}, \bar{b}\bar{b})$ is an eigenvector of $\hat{R}$ with eigenvalue $+q$, we obtain

$$
\bar{a}\bar{b} - q\bar{b}\bar{a} = 0 \qquad \bar{b}^2 = 0 \tag{25}
$$

and by transposition

$$
ab - p^{-1} ba = 0 \qquad b^2 = 0. \tag{26}
$$

Demanding further the associativity of the mixed products of oscillators, we obtain the following two-parameter qp-oscillator algebras

Solution (I): equations (25), (26)

$$\begin{aligned} &a\bar{b} - q^{-1}\bar{b}a = 0 \\ &\bar{a}b - pb\bar{a} = 0 \\ &a\bar{a} - p^{-1}q^{-1}\bar{a}a = 1 \\ &b\bar{b} + \bar{b}b = 1 - (1 - p^{-1}q^{-1})\bar{a}a \end{aligned} \tag{27}$$

Solution (II): equations (25), (26)

$$\begin{aligned} &a\bar{b} - p\bar{b}a = 0 \qquad \bar{a}b - q^{-1}b\bar{a} = 0 \\ &a\bar{a} - pq\bar{a}a = 1 - (1 - pq)\bar{b}b \\ &b\bar{b} + \bar{b}b = 1 \end{aligned} \tag{28}$$

which both reduce to (14), (15) when $p = q$.

We are very grateful to M Dubois-Violette for numerous discussions. We also thank I T Todorov for discussions and suggestions.

References

[1] Kullish P P and Reshetikin N Yu 1989 *Lett. Math. Phys.* **18** 143
Chaichian M and Kullish P 1990 *Phys. Lett.* **234B** 72
Chaichian M, Kullish P and Lukiersky J 1990 *Phys. Lett.* **237B** 401
Hayashi T 1990 *Commun. Math. Phys.* **127** 129
Floreanini R and Vinet L 1990 *J. Phys. A: Math. Gen.* **23** L1019
Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
Floreanini R, Spiridonov V P and Vinet L 1990 *Phys. Let.* **242B** 383; 1991 *Commun. Math. Phys.* **137** 149
[2] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[3] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[4] Pusz W and Woronowicz S L 1989 *Rep. Math. Phys.* **27** 231
Pusz W 1989 *Rep. Math. Phys.* **27** 349
[5] Hadjiivanov L K, Paunov R R and Todorov I T 1992 *J. Math. Phys.* **33** 1379
[6] Chaichian M, Kullish P and Lukiersky L 1981 *Phys. Lett.* **262B** 43
[7] J C Wallet 1992 in preparation
[8] Kac V G 1977 *Commun. Math. Phys.* **53** 31
[9] Dabrowski L and Wang L 1991 *Phys. Lett.* **266B** 51